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Surface profile prediction from bottom pressure measurements with application to marine current generators

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Abstract—**A knowledge of wave kinematics is requisite for most aspects of marine engineering, yet still relatively little is known in the context of wave-current interactions. There is a requirement for wavepower applications to predict the wavefield at a Wave Energy Converter (WEC), particularly for the application of control algorithms. For marine current generators a similar requirement arises. The modelling of wave-current interactions possesses a rich history, yet the presence of vorticity immediately introduces major mathematical complications into modelling considerations. Improving our understanding of the relationship between the dynamic pressure function and the underlying fluid kinematics for rotational ocean waves has implications for both wave and tidal resource characterisation. This paper considers the pressure-streamfunction relationship for a train of regular water waves propagating on a steady current, which may possess an arbitrary distribution of vorticity, in two dimensions. Using a novel pressure-streamfunction reformulation of the governing equations, an explicit formula was recently derived by two of the authors in terms of series solutions detailing the relationship between the pressure, streamfunction and the vorticity distributions. In particular, for linear waves, a description is provided of the role which the pressure function on the sea-bed plays in determining the free-surface profile elevation. These are the first results in this direction for water waves with vorticity, and it is shown that this approach provides a good approximation for a range of current conditions.**

Index Terms—**Pressure recovery, surface prediction, wave-current interactions.**

I. Introduction

A knowledge of wave kinematics is requisite for most aspects of marine engineering. In the context of wavepower applications there is a requirement for wavepower applications to predict the wavefield at a Wave Energy Converter (WEC), particularly for the application of control algorithms and for loading upon the device structure. Often this involves the measurement of the incident wavefield at one or more locations and an approximate method is then employed to predict the incident wavefield in the vicinity of the WEC. This topic is both an interesting and difficult one and one relevant study has previously been reported in EWTEC in [17].

For marine current generators a similar requirement arises, with horizontal and vertical variations in the current being associated with the additional difficulty of the current possessing horizontal and vertical distributions of vorticity. A demonstration of the importance of vorticity to the local kinematics is given in [14].

Accurate measurement of the surface wavefield, from which the wave kinematics may be determined, is not a trivial task. In the absence of a non-uniform current, where the wavefield is irrotational, there is a substantial body of work associated with the measurement of the pressure at or near to the bed and the use of transfer functions to determine the free surface motions (cf. [2], [10], [11]). This is not universally useful, since it depends upon a classification of the wavefield and a reasonable practical correlation between the pressure at the bed and at the free surface. However, in appropriate circumstances it can provide a valuable input into the wave prediction algorithms.

For marine current generators, there is less applicable work but there is the important advantage that most devices are fixed to the sea bed and not restrained by the wave classification difficulty identified above. Flows with vorticity are relevant in several physical contexts and techniques originating in coastal and nearshore engineering may be adapted to be for direct use for the description of the wavefield in the vicinity of marine current generators.

The modelling of wave-current interactions possesses a rich history (cf. [9], [12], [13], [15]), yet the presence of vorticity introduces major mathematical complications into modelling considerations. Improving the understanding of the relationship between the dynamic pressure function and the underlying fluid kinematics for rotational ocean waves has implications for both wave and tidal resource characterization.

This paper considers the pressure-streamfunction relationship for a train of regular water waves propagating on a steady current, which may possess an arbitrary distribution of vorticity, in two dimensions. The application of such work is targeted towards marine current generators. Using a novel pressure-streamfunction reformulation of the governing equations, recently derived by two of the authors in [5], an explicit formula was obtained in terms of series solutions detailing the relationship between the pressure, streamfunction and the vorticity distributions. A benefit of this approach is the construction of explicit formulae for the pressure-transfer function and the pressure amplification factor, relating dynamic pressure measurements at a fixed-depth beneath the wave trough — not necessarily on

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the flat bed — to the linear wave surface elevation, for arbitrary current profiles. These are the first results in this application for waves with vorticity.

The effectiveness of the transfer function formula, shown in (7) in the text for purely irrotational flow, has been widely tested in the engineering literature, through field data and experiments: cf. [3] for an interesting comparison and contrast between a number of different data-sets, for both regular and irregular waves. While some interesting experimental and numerical analyses of the pressure distribution have been undertaken for irregular waves in the literature, for example in [2], [3], [8], this paper will be focussed primarily on regular waves.

Two particular topics addressed throughout the literature are of relevance to the considerations of the current paper. Firstly, one source of speculation for possible discrepancies between theory and observation in the irrotational pressure transfer function is the presence of depth-varying currents, for which the assumption of irrotationality is invalid and vorticity much be included in the fluid model. Secondly, much experimental work has been undertaken (cf. [3]) to establish if the transfer function is sensitive to the relative-depth at which the pressure transducers are located: the pressure transfer formula for irrotational waves may be applied regardless of the depth at which the pressure is measured and, in particular, pressure sensors need not be located on the sea-bed.

The formulae for waves with vorticity, derived in [5], are similarly unrestricted with regard to the exact location of the pressure measurements, and it is a reference pressure level that is required. These formulae are highly suited to the Moderate Current Approximation regime, described herein, where they assume a particularly amenable form. The aim of this paper is to validate this new model through testing these formulae numerically for a range of underlying current distributions considered to be typical wave-current interactions for both adverse and following currents. In the process it is established that the formulae provide good approximations for a wide range of currents and at a range of reference pressure levels.

II. Modelling wave-current interactions

Marine currents can be regarded as being locally constant when compared to the time variations associated with the local wave-field or turbulence in the flows [14]. Accordingly, the local velocity field below the wave trough can be expressed as

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{U}(x,y,z)+\boldsymbol{u_w}(x,y,z,t),$$

where $\langle \boldsymbol{u_w}(x,y,z,t)\rangle = 0$. Here $\langle\cdot\rangle$ denotes the time average, which for regular waves is taken over a single period, while a longer measure should be employed for irregular wave fields. A requisite of attempting to predict the wave-current kinematics is that the underlying current profile is either assumed known *a priori*, or can be obtained to the required degree of accuracy; such methods employ either numerical predictions or fieldwork.

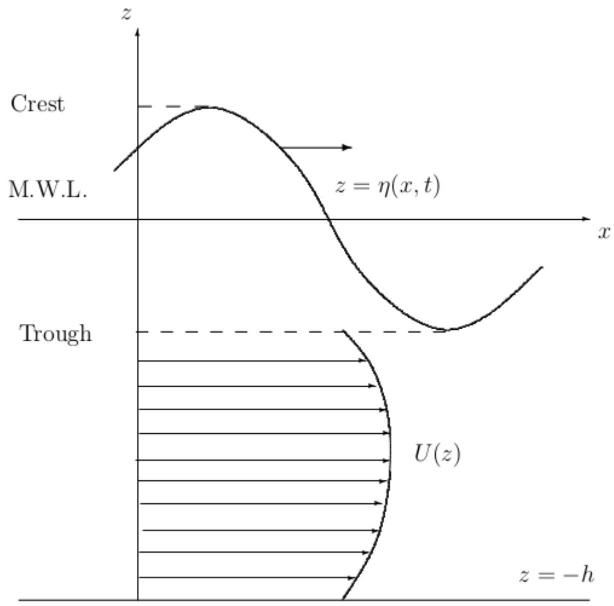

Fig. 1. Schematic of vertical section for two-dimensional wave-current interactions.

It is usual to choose the x-axis so that it is locally aligned with the principal horizontal direction of the current, with the z-axis then being measured positive in a vertically upwards direction. The y-axis then completes the right-handed coordinate system. In this paper we focus on two-dimensional wave-current interactions which are confined to a vertical plane. A schematic is shown in Fig. 1 in which the current takes the form $\mathbf{U}=(U(z),0,0)$. From a modelling perspective, it is usual to assume that the density is constant and that the water is incompressible. Unless attention is directed towards the narrows viscous boundary layer at the bed, then it is reasonable to assume that viscosity is unimportant. It is further assumed that spatial periodicity is present with respect to the x-variable. The origin O is chosen to lie in the mean water level: if $z=\eta(x,t)$ represents the unknown free-surface, and $\lambda>0$ is the characteristic wavelength, then

$$\int_0^\lambda \eta(x,t)dx=0.$$

The choice of reference frame implies that $z=-h$ denotes the location of the impermeable flat bed, which is assumed to be locally horizontal and where h is the mean water depth.

III. Governing equations

The equations of motion comprise the incompressibility equation

$$u_x+w_z=0, \tag{1a}$$

while the Euler equation takes the form

$$\begin{aligned} u_t+uu_x+wu_z&=-\frac{p_x}{\rho},\\ w_t+uw_x+ww_z&=-\frac{p_z}{\rho}-g, \end{aligned} \tag{1b}$$

where u, w are the horizontal and vertical velocities respectively, p represents the pressure distribution, ρ is the fluid density (assumed to be constant) and g is the standard gravitational constant of acceleration. On the flat bed the kinematic boundary condition gives

$$w = 0 \qquad \text{on } z = -h, \tag{1c}$$

and at the free-surface the kinematic and dynamic conditions are

$$w = \eta_t + u\eta_x, \tag{1d}$$
$$p = \text{constant} \qquad \text{on } z = \eta(x,t). \tag{1e}$$

Typically, the constant atmospheric pressure, denoted p_a, is taken as the free-surface value in (1d) and the pressure can be written as

$$p = p_a - \rho g z + p_d(x,z,t), \tag{2}$$

where p_d denotes the dynamic pressure, measuring the deviation from the hydrostatic pressure. The nonlinear free-boundary value problem specified by the system of equations (1) represents the full governing equations for water waves in two dimensions. The vorticity prescribed by fluid motion is defined as the curl of the fluid velocity field, $\mathbf{\Omega} = \nabla \times \boldsymbol{u}$, which reduces in two-dimensions to $\mathbf{\Omega} = (0, \Omega, 0)$, leading to the scalar vorticity equation

$$\Omega(x,z,t) = u_z - w_x. \tag{3}$$

IV. Surface Profile Prediction: Irrotational Wave Motion

Following a standard linearisation procedure and assuming the surface profile to be

$$\eta(x,t) = a\cos(kx - \omega t) \tag{4}$$

for a regular linear wave solution, the linearised form of the governing equations (1) may be solved explicitly for waves satisfying the irrotationality condition $\Omega \equiv 0$, leading to

$$p_d(x,z,t) = \rho a g \frac{\cosh k(z+h)}{\cosh kh}\cos(kx - \omega t). \tag{5}$$

Here a is the wave amplitude, $k = 2\pi/\lambda$ is the wavenumber and ω is the phase frequency; these quantities are not independent but are related by $c = \omega/k$, where c is the wave phasespeed. In the present setting of linear waves over a flat bed it may be shown that the dispersion relation $\omega^2 = gk\tanh kh$ holds, leading to the prescription of the wave phase-speed as

$$c = c(k) = \sqrt{g\tanh kh/k}. \tag{6}$$

Comparing (4) and (5) leads to the so-called transfer function formula

$$\eta(x,t) = \frac{p_d(x,z,t)}{\rho g K_p(z)} \tag{7}$$

where $K_p(z) = \cosh k(z+h)/\cosh kh$ is sometimes called the pressure response factor. Note that this relationship may seem to be inconsistent as the right-hand-side depends upon three independent variables whereas the left-hand-side depends only upon two; however, the left-hand-side is independent of z by construction.

The effectiveness of the transfer function formula (7) has been widely tested through field data and experiments; for example, cf. [3] for an interesting comparison and contrast between a number of different data-sets, for both regular and irregular waves. Among the issues considered in [3] is the need to offset any potential inaccuracies between theory and observations by multiplying the right-hand side of (7) with an empirical correction factor N, a constant which may vary depending on the local environment to which the particular data set relates. The authors outline the differing opinions which prevail regarding the necessity for, and possible behaviour of, this empirical correction factor. They conclude that such a factor is probably unnecessary (that is, it is reasonable to choose $N = 1$) with any perceived discrepancies between theory and observation being accounted for by issues such as inaccurate measurements, instrument limitations and analysis methods. Furthermore, they assert that "linear theory is adequate to compensate pressure data and give reliable estimates of surface wave heights" for most of the wave amplitudes considered.

Further experimental studies of interest regarding the transfer function are found in [7], where the authors consider a purely empirical expression for the transfer function derived through dimensional-analysis considerations; it is shown in [1] that this formula may be regarded as a version of the theoretically derived formula (7). Further analysis and discussion of these issues was undertaken in the more recent paper [16].

The considerations involved in the derivation of (4)–(6), and the subsequent pressure recovery formulae, may be readily adapted to accommodate an irrotational velocity field of the form $(u+U, 0, w)$, where $U(z) \equiv U$ is a constant underlying current, cf. [9], [11], [15]. In so doing, the 'absolute' wave frequency ω must be directly replaced by the 'relative' or 'intrinsic' wave frequency $\sigma := \omega - kU$: the presence of a constant underlying current amounts to a form of Doppler shift in the wave motion. These modifications have a number of interesting ramifications, not least that it may be discerned directly from the modified dispersion relation $\sigma^2 = gk\tanh kh$ as to which wave motions, depending on the magnitude and direction (adverse or following) of the current, are admissible, cf. [9].

An additional physical quantity, which is of practical significance, is the pressure amplification factor Q, relating the dynamic pressure at the free surface to the dynamic pressure of the fluid at any arbitrary depth $-h \le z < 0$. Expressing $p_d(x,z,t) = P_d(z)\cos(kx - \omega t)$,

$$Q(z) = \frac{P_d(0)}{P_d(-z)}. \tag{8}$$

The pressure response factor $K_p(z)$ is related to the amplification factor by way of $Q(z) = 1/K_p(z)$. It is straightforward to show that $Q(0) = \cosh kh$ and this imposes a limit on the usefulness of this approach in terms of the wave regime. For deep water waves, defined by $kh >> 1$, the bed pressure will be very small in comparison to the surface pressure and the process will not provide a useful method for surface prediction; inaccuracies in the pressure measurement will be magnified throughout the water column. Thus the method will work best for shallow wave waves and decrease in effectiveness if kh increases towards the deep water regime. In a physical context, this means that is likely to be more efficient generally in the coastal zone than in an oceanographic environment.

V. Reformulation

Recently, a new pressure–streamfunction formulation of the governing equations (1) for rotational flows was constructed in [5], the salient features of which are presented. Defining the stream function $\Psi(x,z,t)$ by

$$u = \frac{\partial \Psi}{\partial z}, \quad w = -\frac{\partial \Psi}{\partial x}, \tag{9}$$

it follows immediately from (3) and (9) that

$$\nabla^2 \Psi = \Omega, \tag{10}$$

where ∇^2 represents the (two-dimensional) Laplacian operator. Focussing attention on regular wave solutions, by which we mean that the x,t dependence can only occur via a phase function $\theta(x,t) = kx - \omega t$, re-expresses the $x-$ and $t-$derivatives as

$$\frac{\partial}{\partial x} = k\,\frac{\partial}{\partial \theta}, \qquad \frac{\partial}{\partial t} = -\,\omega\,\frac{\partial}{\partial \theta}. \tag{11}$$

With an absence of detail, it can be shown that

$$\begin{aligned} \frac{1}{\rho}\frac{\partial p}{\partial \theta} &= \frac{\partial H}{\partial \theta} - \Psi\,\nabla^2 \Psi_\theta, \\ \frac{1}{\rho}\frac{\partial p}{\partial z} &= \frac{\partial H}{\partial z} - \frac{\omega}{k}\nabla^2\Psi - \Psi\,\nabla^2\Psi_z, \end{aligned} \tag{12}$$

where

$$H(\theta,z) = -gz + \frac{\omega}{k}\Psi_z - \frac{1}{2}\left\{\Psi_z^2 + k^2\Psi_\theta^2\right\} + \Psi\,\nabla^2\Psi.$$

The terms on the right-hand side of (12), which are not included in $H(\theta,z)$ are strictly rotational terms, where $H(\theta,z)$ represents a form of integrability in the equations. Elimination of $p\,(\theta,z)$ in (12) is straightforward and yields an equation in Ψ, which is the streamfunction method. However, this approach is not adopted at this stage as the principle interest is directed towards the pressure. In order to seek analytic or semi-analytic solutions in the pressure-streamfunction formulation (12), the streamfunction and the pressure-type terms are represented as

$$\begin{aligned} \Psi(\theta,z) &= \sum_{n=0}^{\infty} \psi_n(z)\cos n\theta, \\ p(\theta,z) &= -\rho g z + \sum_{n=0}^{\infty} p_n(z)\cos n\theta, \end{aligned} \tag{13}$$

and the vorticity $\Omega (= \nabla^2\Psi)$ is also expressed in the same way,

$$\Omega\,(\theta,z) = \sum_{n=0}^{\infty} \Omega_n(z)\cos n\theta,$$

Here p is now the pressure relative to the atmospheric pressure. It is known that analytic solutions have been obtained only for the special cases of $\Omega_0 = 0$ or Ω_0 =constant, both corresponding to irrotational wave motions. Also, from (12) and (13), the hydrostatic component of the pressure is identified explicitly and need not be included in the subsequent analysis. With the hydrostatic component omitted the full dynamic pressure-streamfunction relationship can be expressed

$$\begin{aligned} \frac{1}{\rho}\sum_{n=0}^{\infty} p_n(z)\cos n\theta &= -A_0 - \frac{1}{2}\left(\psi_0'(z)\right)^2 + \int^z \psi_0'(z)\Omega_0(z)dz \\ &- \frac{1}{4}\sum_{m=1}^{\infty}\left[\left(\psi_m'(z)\right)^2 + \left(mk\psi_m(z)\right)^2 - 2\int^z \psi_m'(z)\Omega_m(z)dz\right] \\ &+ \sum_{n=1}^{\infty}\left\{\frac{\omega}{k}\psi_n'(z) - \frac{1}{2}\left[\psi_n'(z)\psi_0'(z) - 2\psi_n(z)\Omega_0(z)\right]\right\}\cos n\theta \\ &- \frac{1}{4}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[\psi_n'(z)\psi_m'^2\psi_n(z)\psi_m(z) - \frac{2n}{m+n}\psi_n(z)\Omega_m(z)\right]\cos(n+m)\theta \\ &- \frac{1}{4}\sum_{n=0}^{\infty}\sum_{\substack{m=1 \\ m\neq n}}^{\infty}\left[\psi_n'(z)\psi_m'^2\psi_n(z)\psi_m(z) + \frac{2n}{m-n}\psi_n(z)\Omega_m(z)\right]\cos(m-n)\theta, \end{aligned} \tag{14}$$

where the identity $\Omega_0 = \psi_0''$ has been employed. This relation is valid up to constants of integration, which vanish identically since $\psi_n(-h) = 0$ for all $n \geq 0$. The first order terms in (13) capture linear wave motion and wave-current interactions. If the current takes the form $(U(z),0,0)$, then from (9) and (13), $\psi_0(z)$ can be easily determined to be

$$\psi_0(z) = \int_{-h}^{z} U(z)dz. \tag{15}$$

At first order, the assumption that $\psi_q(z)$ is of $\mathcal{O}\left(\epsilon^q\right)$ yields the following equation for $\psi_1(z)$:

$$\psi_1''(z) - \left(k^2 - \frac{kU''(z)}{\omega - kU(z)}\right)\psi_1(z) = 0. \tag{16}$$

For regular waves over a horizontal bed with mean water depth h, the appropriate boundary conditions at the bed $z=-h$ and surface $z=0$ are

$$(c-U(0))^2\psi_1'(0) + \left[(c-U(0))\,U'(0) - g\right]\psi_1(0) = 0,$$

$$\psi_1(0) = a\,(c - U(0))\,, \qquad (17)$$
$$\psi_1(-h) = 0,$$

where a is the wave amplitude, defined to be the magnitude of the first harmonic of the surface elevation series, and $c = \omega/k$ is the phase velocity. In the derivation of the surface conditions the usual assumptions of linearity apply. This is the Rayleigh equation of hydrodynamic stability theory, or the inviscid form of the Orr-Sommerfeld equation. Particularly in the context of considering the pressure distribution, it is worth noting that an alternative formulation to (16), expressed in terms of $p_1(z)$ rather than $\psi_1(z)$, may be employed for linear solutions [9].

The first harmonic of dynamic pressure $p_1(z)$ is obtained from (14), with the appropriate ordering implemented and (15) utilised, resulting in

$$\frac{p_1(z)}{\rho} = \left(\frac{\omega}{k} - U(z)\right)\psi_1'(z) + U'(z)\psi_1(z). \qquad (18)$$

Bearing in mind (13), the decomposition (2) and the standard linear wave profile (4), $p_1(z)$ must also satisfy

$$p_1(0) = \rho g a \qquad (19)$$

to ensure that condition (1d) holds. A noteworthy consequence of (19) is that $p_n(0) = 0$ for the higher order ($n \geq 2$) dynamic pressure terms in (13). Thus, if $\psi_1(z)$ can be determined from (16) and (17), then $p_1(z)$ can be obtained from (18).

For the pressure recovery problem, the aim is to measure the dynamic bed-pressure $p_b = p_1(-h)$ and use this to determine the surface amplitude a. In this formulation, the opposite approach is taken, namely that a is used to determine $\psi_1(z)$ and hence the pressure via (18). However, $\psi_1(z)$ is linearly proportional to a, as can be seen from (16) and (17), and thus

$$\psi_1(z) = a\chi_1(z), \qquad (20)$$

for an appropriate function $\chi_1(z)$ and from (18),

$$\begin{aligned} p_b = p_1(-h) &= \rho\,[c - U(-h)]\,\psi_1'(-h) \\ &= \rho a\,[c - U(-h)]\,\chi_1'(-h), \end{aligned} \qquad (21)$$

as $\psi_0(-h) = \psi_1(-h) = 0$ from (15) and (17). Thus if p_b is the known quantity, then

$$a = \frac{p_b}{\rho\,[c - U(-h)]\,\chi_1'(-h)}. \qquad (22)$$

The amplification factor $Q(-h)$, defined in (8), relating the pressure at the free surface to the pressure at the bed now follows directly from (19) and (21):

$$Q(-h) = \frac{g}{[c - U(-h)]\,\chi_1'(-h)}. \qquad (23)$$

This formula is the generalisation of the amplification factor to the setting of water waves with general vorticity distributions; the formula for irrotational waves is recovered immediately upon setting $U(z) \equiv U$, where U is a constant underlying current. This is easily verified by taking $U = 0$, using the dispersion relation (6) for the wave phase speed c and explicitly determining $\chi_1'(z)$. Furthermore, it is clear that there is no restriction to working at the flat bed ($z = -h$) in taking pressure measurements, although matters are slightly more complicated otherwise; indeed, an alternative choice such as $Q(z_r) = p_1(0)/p_1(z_r)$ may be made, working again with (19) and inputting an arbitrary depth $-h \leq z_r < 0$ in formula (18), leading to a generalisation of (21) and (23).

The drawback of the described approach in using (18) to analyse the relationship between the dynamic pressure and the wave-field kinematics is that, for arbitrary $U(z)$, (16) and (17) can only be solved numerically (as described in [12]) and for this reason approximate solutions are required in general. Nevertheless, in the special cases of a uniform underlying current $U(z) = U_c$ (zero vorticity), and also for constant vorticity distributions (representing a linearly–sheared current profile $U(z) = U_s + \Omega z$), explicit solutions can be derived, cf. [15], in which case the recovery formula (23) is applicable. Accordingly, it is worthy of mention that many current profiles can be approximated by a number of linear components and with appropriate matching conditions applied at the interfaces.

A. Moderate Current Approximation

Although numerical solutions are not difficult to obtain at this order, the lack of an analytic solution prevents simple insights to be gained and also hinders progress to higher orders. For this reason a perturbation approach is developed for weakly nonlinear waves, consistent with Stokes waves for irrotational wave motion and with the pressure–streamfunction formulation presented above. In contrast to most procedures in water waves, two non-dimensional perturbation parameters are employed — the wave slope ε, already utilised above, and δ as a measure of the current strength relative to the phase speed of the waves: typically $\delta = \hat{U}/c$, with $\hat{U}$ a characteristic current measure. The presence of δ is formally recognised by writing

$$U(z) = \delta V(z)\,, \qquad (24)$$

so that $V(z)$ has the same dimensions as the current but is of comparable magnitude to the wave phase speed c. There is no requirement to evaluate δ explicitly, it is simply a measure to indicate relative magnitude. A full history of this approach is provided by [15].

Perturbation solutions for the streamfunction Ψ, surface elevation η and pressure p are sought of the form

$$\begin{aligned} \Psi &= \Psi_{00} + \delta\,\Psi_{01} + \delta^2\,\Psi_{02} + \cdots + \varepsilon\,(\Psi_{10} + \delta\,\Psi_{11} + \cdots) + \varepsilon^2\,(\Psi_{20} + \delta\,\Psi_{21} + \cdots) + \ldots \\ \eta &= \eta_{00} + \delta\,\eta_{01} + \delta^2\,\eta_{02} + \cdots + \varepsilon\,(\eta_{10} + \delta\,\eta_{11} + \cdots) + \varepsilon^2\,(\eta_{20} + \delta\,\eta_{21} + \cdots) + \ldots \\ p &= -\rho g z + p_{00} + \delta\,p_{01} + \delta^2 p_{02} + \cdots + \varepsilon\,(p_{10} + \delta\,p_{11} + \cdots) + \varepsilon^2\,(p_{20} + \delta\,p_{21} + \cdots) + \ldots \end{aligned} \qquad (25)$$

An intuitive interpretation of the representation defined by (25) is that the formal setting $\delta = 0$ removes the imposed current but permits mean flows associated with the waves, depending upon the choice of reference frame. Similarly the setting $\varepsilon = 0$ corresponds to the case of a current alone in the absence of waves and mixed terms of O($\varepsilon^i \delta^j$) describe the interaction terms. To maintain consistency with established practice in Stokes wave theory, it is also necessary to expand the frequency ω of the waves (or phase speed c) in a similar manner,

$$\omega = \omega_{00}+\delta\omega_{01}+\delta^2\omega_{02}+\cdots+\varepsilon\left(\omega_{10}+\delta\omega_{11}+\cdots\right)+..., \quad (26)$$

and this is only meaningful when ε is non-zero.

The scheme in (25) and (26) requires the imposition of a relative ordering upon ε and δ. Thomas & Klopman [15] proposed a classification scheme of three regimes based upon the relative magnitude of the parameters, together with a discussion of the salient issues involved. In the terminology of that paper, the Moderate Current Approximation (MCA) is defined by the regime $\varepsilon << \delta << 1$ and is the one of interest here. It is noted that the Strong Current Approximation defined by $\varepsilon << 1$, $\delta \sim O(1)$ is the same as the one employed linearly in previous considerations and requires numerical evaluation.

Implementation involves substituting the series for Ψ, η and ω into the Helmholtz equation obtained from combining (12) and formulating a hierarchy in the usual manner. It is hoped that these can be solved when appropriate boundary conditions are applied and p can then be obtained from (12). As mentioned previously, the problem can be formulated in terms of the pressure at $O\left(\varepsilon\right)$ [9] but this cannot be readily extended beyond $O\left(\varepsilon\right)$ and thus the streamfunction Ψ is employed here. The basic current term is at $O\left(\delta\right)$ and is given by

$$\Psi_{01}(z) = \int_{-h}^{z} V(z)\, dz \; . \quad (27)$$

The first wavelike term at $O\left(\varepsilon\right)$ is the incident wave in the absence of a current. As the streamfunction component $\Psi_{10}(z,\theta)$ satisfies $\nabla^2\Psi_{10} = 0$ and is the known linear wave solution we have

$$\eta_{10} = \frac{1}{k}\cos\theta(x,t), \;\; \Psi_{10} = \frac{\omega_{00}}{k^2}\,\frac{\sinh k(z+h)}{\sinh kh}\,\cos\theta(x,t),$$
$$p_{10} = \rho\frac{\omega_{00}}{k}.\frac{\partial\Psi_{10}}{\partial z}, \quad (28)$$

with

$$\omega_{00}^2(k) = gk\tanh kh.$$

If kinematic evaluation is required, then the wave slope $\varepsilon = ak$ must be included as in (25). The primary interaction term occurs at $O\left(\varepsilon\delta\right)$ and is the MCA equivalent of (18) correct to $O\left(\varepsilon\delta\right)$. The derivation of all the $O\left(\varepsilon\delta\right)$ terms is given in the *Appendix* and the results will be retrieved here as necessary.

The wavelike pressure correct to $O\left(\varepsilon\delta\right)$ is the sum of p_{10} and p_{11}. These terms are given in (28) and (35) respectively; for convenience this combined pressure is written just as $p\left(\theta,z\right)$. As the $\theta-$dependency is the same in all terms, it is convenient to write the pressure as

$$p\left(\theta,z\right) = \rho a\;\Pi\left(z\right)\cos\theta\;. \quad (29)$$

A similar analysis to that conducted in the earlier part of this section may now be undertaken, with the difference being that $\Pi\left(z\right)$ is known and can be evaluated. If the bed and surface pressures are written as

$$p\left(\theta,-h\right) = \rho a\;\Pi\left(-h\right)\cos\theta = p_b\cos\theta,$$
$$p\left(\theta,0\right) = a\;\Pi(0)\cos\theta,$$

then the surface elevation a is

$$a = \frac{p_b}{\rho\Pi\left(-h\right)}\;, \quad (30)$$

and the pressure amplification factor Q is

$$Q = \frac{\Pi\left(0\right)}{\Pi\left(-h\right)}\;. \quad (31)$$

In contrast to the corresponding expressions for a and Q in (22) and (23), all terms in (30) and (31) are known and can be evaluated for a given current profile $U\left(z\right)$. Also, with $\Pi(-h)$ obtained from (36) and (29) with $z = -h$, the last term in the expression is zero since $\Psi_{10}\left(\theta,-h\right) = 0$ by construction. Furthermore, in common with the considerations of previous sections there is no restriction at the depth at which reference pressure measurements are taken. Thus for a reference pressure taken at $z = z_r$, where $h \le z_r < 0$, the choice corresponds to $Q(z_r) = \Pi\left(0\right)/\Pi\left(z_r\right)$. This is a useful approach as it extends the validity to a measurement above the bed.

VI. Results

There are no measurements from a dedicated experimental programme against which to test the predications of this general approach; nor are field measurements readily available. Thus the existence of models associated with particular cases and the availability of numerical studies has become important.

Two particular approaches were utilised. The first employs the analytic formulae presented by Brink-Kjaer [4] for the case of a linear current profile and which provide expressions for the pressure as well as the usual kinematic quantities. Although these are presented to second order, only the linear case is employed to retain consistency with the present work. The idea is to stipulate the parameters of a linear current profile given by $U\left(z\right) = U_0+\Omega z$, i.e. the surface value U_0 and the constant vorticity Ω, and then input the pressure at the bed p_0 for a given wave amplitude; the MCA pressure prediction model is then used with p_b as input to predict the amplitude. This enables a comparison to be made between prediction and known form, by way of a relative error, for a range of U_0 and Ω. The second approach utilises numerical results that are already available from [15] for a current with a specified depth variation.

The profiles have been selectively chosen so as to be representative of an adverse and a following current, as described more fully in [15], with data taken from

Klopman [6]. The considered set-up is for a system of water depth $0.5m$, wave amplitude $0.005m$ and angular frequency ω is $1.6\pi\, rad\, s^{-1}$, as described in [15]. All calculations were completed using MATLAB.

Four current profiles are considered. The first is

$$U_1(z) = (-0.261 - 0.359z)ms^{-1},$$

which is an adverse profile with constant vorticity fitted to experimental data from [4]. The second profile is

$$U_2(z) = (-0.4 - \Omega z)ms^{-1},$$

structured to be a general adverse profile for which the vorticity Ω can be varied. Profile three is

$$U_3(z) = -0.20 \exp{(2.778z)}\, ms^{-1}$$

and is a test adverse arbitrary profile used in [15]. The final profile $U_4(z)$ is a following profile fitted to experimental data using 10th order Chebyshev orthogonal polynomials of the first kind. The experimental data is taken from [6].

Figure 2 shows the profiles and the experimental data upon which two are based. As data is not available for the uppermost layer, above a depth of $0.1m$ for U_4, a rational polynomial fit has been applied to the five highest points and extrapolated to the MWL as shown by the dashed line.

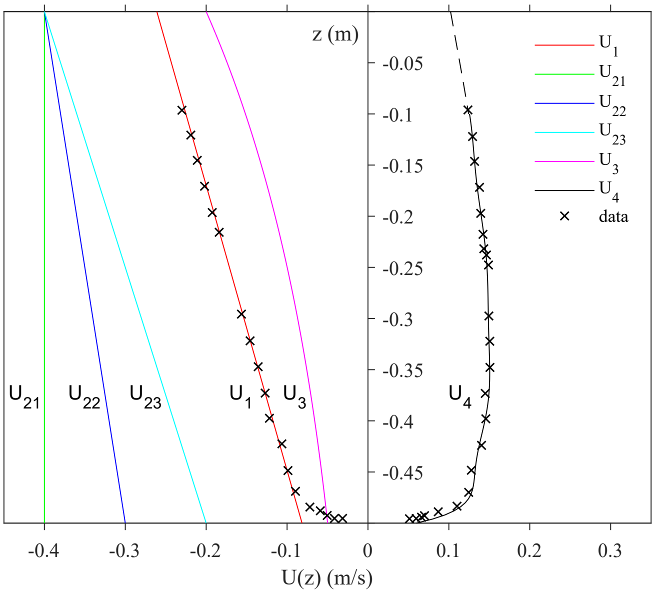

Fig. 2. Current profiles. For U_{21}, U_{22} and U_{23} the vorticities are 0, -0.2 and -0.4s^{-1} respectively.

For a given current and the appropriate physical variables, such as water depth and wave frequency, the method of prediction of surface amplitude is as follows. The first step is to solve the dispersion relation (34) to obtain the wavenumber k, then determine Ψ_{11} from (32) and p_{11} from (35) to obtain the pressure amplifications from (30) and (31). For two special cases it is possible to monitor progress prior to final evaluations. The profile U_3 was employed in [15] and the accuracy is confirmed in Table 1 of that paper. For the linear profile U_2, Table I below shows the wavenumber predictions for three cases of U_2, with each corresponding to a different vorticity value Ω. It is clear from this table that the differences in k from the MCA and the appropriate exact dispersion relation are insignificant.

For the current profiles U_1 and U_2, which possess constant vorticity, the dynamic pressure values are determined following the strategy outlined above from the velocity distribution following [4] where the amplitude is known. The amplitude is then recovered using (30) where these dynamic pressure values are the inputs. The approach initially requires solving the dispersion relation (34) to obtain the wavenumber. Wavenumber calculations compared with those obtained using the approach in [4] are shown in Table I. In the case of zero vorticity (U_{21}) the results correspond exactly but increasing the vorticity (U_{22} and U_{23}) introduces a small discrepancy.

TABLE I
WAVENUMBER COMPARISON.

Profile	Brink-Kjær	MCA	Difference
U_{21}	5.20896	5.20896	0.000%
U_{22}	4.91387	4.91586	0.040%
U_{23}	4.64248	4.64991	0.160%

The amplitude recovery procedure is initially applied to profile $U_1(z)$. Table II shows the recovered amplitude, which is to be compared to the target value of 0.005, together with the pressure amplification factor Q. The pressure reference values are determined from [4] and are given in Pascals. It is clear that the variations between the target and predicted amplitudes are insignificant. As can be seen in Table II, noting that pressure measurements are given in Pascals, the recovered amplitude matches the expectation very closely (at the scale of microns). The amplification factor Q is also displayed. The variation in Q shows the attractiveness of moving away from the bed if possible.

TABLE II
AMPLITUDE RECOVERY $U_1(z) = -0.261 - 0.359z$.

Depth	$p(z)$	Recovered a	Difference	$Q(z)$
-0.50	14.816	0.00500035	0.007%	3.311
-0.40	15.811	0.00499988	-0.002%	3.103
-0.30	18.993	0.00499916	-0.017%	2.583
-0.20	24.894	0.00499889	-0.022%	1.970
-0.10	34.435	0.00499923	-0.015%	1.424

Tables III - V employ the the profile U_2 for differing values of the vorticity Ω, with an initial value of $\Omega = 0$. These tables confirm the comments made above with regard to the position of the reference level and the accuracy of prediction. The first table in this sequence (Table III) corresponds to a constant current and so very good agreement should be expected. There is however, some increase in error with an increase in vorticity, although it is insignificantly small.

TABLE III
AMPLITUDE RECOVERY $U_{21} = -0.4$.

Depth	$p(z)$	Recovered a	Difference	$Q(z)$
-0.50	7.216	0.00500000	0.000%	6.799
-0.40	8.217	0.00500000	0.000%	5.971
-0.30	11.498	0.00500000	0.000%	4.267
-0.20	17.971	0.00500000	0.000%	2.730
-0.10	29.431	0.00500000	0.000%	1.667

TABLE IV
AMPLITUDE RECOVERY $U_{22}(z) = -0.4 - 0.2z$.

Depth	$p(z)$	Recovered a	Difference	$Q(z)$
-0.50	7.986	0.00500180	0.036%	6.146
-0.40	8.979	0.00500142	0.028%	5.465
-0.30	12.248	0.00500078	0.016%	4.006
-0.20	18.676	0.00500034	0.007%	2.627
-0.10	29.961	0.00500011	0.002%	1.637

TABLE V
AMPLITUDE RECOVERY $U_{23}(z) = -0.4 - 0.4z$.

Depth	$p(z)$	Recovered a	Difference	$Q(z)$
-0.50	8.751	0.00500600	0.120%	5.613
-0.40	9.731	0.00500467	0.093%	5.046
-0.30	12.973	0.00500243	0.049%	3.784
-0.20	19.341	0.00500091	0.018%	2.537
-0.10	30.449	0.00500023	0.005%	1.611

Table VI shows the pressure amplitude predictions for profiles $U_3(z)$ and $U_4(z)$. These cannot be compared with other models and provide the process for prediction when working with laboratory or field data. It is clear, once more, that from practical and theoretical considerations, it is desirable to move the the point of reference pressure measurement away from the bed.

TABLE VI
AMPLIFICATION FACTOR FOR $U_3(z)$ AND U_4(Z).

Depth	$Q(z)$ for U_3	$Q(z)$ for U_4
-0.50	2.827	1.982
-0.40	2.677	1.918
-0.30	2.295	1.744
-0.20	1.822	1.504
-0.10	1.373	1.245

VII. CONCLUSION

This paper establishes a methodology for predicting the amplitude of surface waves from the measurement of the pressure at the bed or at an appropriate reference level, with an emphasis being placed upon measurement away from the bed if possible. Initial findings, based upon comparison with existing work are very encouraging - showing good accuracy and relative ease of use.

The model is a preliminary one, extending previous work involving waves alone to include a steady current possessing arbitrary variation with depth. In this instance it is assumed that there is a single frequency but this model can be extended to include an incident defined by a linear sprectrum. In its present form it is suitable for application to determine the wavefield in the vicinity of a marine current generator.

One of the required inputs is the ambient current $U(z)$ and it is readily acknowledged that obtaining the current profile is not a simple task. Recent advances in the design and application of ADCPs is very encouraging and this gives some hope that the method can be usefully employed. Within the context of wavepower devices and marine current generators this approach does not solve the complete prediction problem, rather it provides a description of the incident wavefield at a point for input into other models to predict the wavefield at or within the vicinity of the device. It should be stressed that this last step is itself an unresolved problem.

APPENDIX A

The streamfunction component Ψ_{11} has previously been obtained in [15] and presented here in a slightly different form. If $S(z), C(z), S2(z)$ and $C2(z)$ denote $\sinh k(z+h)$, $\cosh k(z+h)$, $\sinh 2k(z+h)$ and $\cosh 2k(z+h)$ respectively, and the functions $I_s(z)$ and $I_c(z)$ are defined by

$$I_s(z) = \int_{-h}^{z} V(z) \sinh 2k(z+h)\, dz\,,$$
$$I_c(z) = \int_{-h}^{z} V(z) \cosh 2k(z+h)\, dz\,,$$

then if $\Psi_{11}(\theta, z)$ is written as $\widetilde{\Psi}_{11}(z)\cos\theta(x,t)$, then we can solve for $\widetilde{\Psi}_{11}(z)$ to get

$$\widetilde{\Psi}_{11} = -\frac{S(z)}{S(0)}.\frac{V(z)}{k} + \frac{2}{S(0)}.\left[I_c(z)C(z) - \frac{C2(0)}{S2(0)}.I_c(0)S(z)\right] + 2\frac{S(z)}{S(0)}.\left[I_s(0) - I_s(z)\right]. \tag{32}$$

This solution necessitates that the unknown function $\omega_{01}(k)$ satisfies

$$\omega_{01}(k) = \frac{2k^2}{\sinh 2kh}\int_{-h}^{0} V(z)\cosh 2k(z+h)dz. \tag{33}$$

Thus from (26), the dispersion relation at this order is

$$\omega = \omega_{00}(k) + \delta\,\omega_{01}(k) + ... =$$
$$\sqrt{gk\tanh kh} + \delta\,\frac{2k^2}{\sinh 2kh}\int_{-h}^{0} V(z)\cosh 2k(z+h)dz + ...$$

In terms of the physical current $U(z)$ and correct to the order of working, this can also be written as

$$(\omega - k\widetilde{U}(k))^2 = gk\tanh kh, \tag{34}$$

with

$$\widetilde{U}(k) = \frac{2k}{\sinh 2kh}\int_{-h}^{0} U(z)\cosh 2k(z+h)dz.$$

This process enables k to be determined once ω, h and $U(z)$ are specified. To obtain the pressure, it is necessary to take the perturbation respresentation for Ψ and p from (25) and (26) and substitute into (12) or (14). Comparison of the appropriate $\varepsilon^m\delta^n$ combination will give p_{mn} in terms of the Ψ_{mn}. At $O(\varepsilon\delta)$, the two equations from (12), relating equations relating p_{11}, Ψ_{11} and Ψ_{10} are

$$
\begin{aligned}
\frac{1}{\rho}\frac{\partial p_{11}}{\partial\theta} &= \frac{\partial H_{11}}{\partial\theta} - \Psi_{01}\,\nabla^2\left(\frac{\partial\Psi_{10}}{\partial\theta}\right) - \Psi_{10}\,\nabla^2\left(\frac{\partial\Psi_{01}}{\partial\theta}\right),\\
\frac{1}{\rho}\frac{\partial p_{11}}{\partial z} &= \frac{\partial H_{11}}{\partial z} - \frac{1}{k}\left[\omega_{00}\nabla^2\Psi_{11} + \omega_{01}\nabla^2\Psi_{10} + \omega_{10}\nabla^2\Psi_{01}\right] - \Psi_{01}\,\nabla^2\left(\frac{\partial\Psi_{10}}{\partial z}\right)\\
&\quad -\Psi_{10}\,\nabla^2\left(\frac{\partial\Psi_{01}}{\partial z}\right),\\
H_{11}(\theta,z) &= \frac{1}{k}\left\{\omega_{00}\frac{\partial\Psi_{11}}{\partial z} + \omega_{01}\frac{\partial\Psi_{10}}{\partial z} + \omega_{10}\frac{\partial\Psi_{01}}{\partial z}\right\} - \frac{\partial\Psi_{01}}{\partial z}.\frac{\partial\Psi_{10}}{\partial z} + k^2\,\frac{\partial\Psi_{01}}{\partial\theta}.\frac{\partial\Psi_{10}}{\partial\theta}\\
&\quad +\Psi_{01}\,\nabla^2\Psi_{10} + \Psi_{10}\,\nabla^2\Psi_{01}\,.
\end{aligned}
$$

Considerable simplification is possible, as Ψ_{01} is a function of z, from (26), and $\Psi_{10}(\theta,z)$ satisfies Laplace's equation. It is straightforward, though tedious, to show that the two equations possess the solution

$$
\begin{aligned}
\frac{1}{\rho}p_{11}(\theta,z) = \frac{1}{k}\left\{\omega_{00}\frac{\partial\Psi_{11}}{\partial z} + \omega_{01}\frac{\partial\Psi_{10}}{\partial z}\right\} - \frac{\partial\Psi_{01}}{\partial z}.\frac{\partial\Psi_{10}}{\partial z}\\
+k^2\,\frac{\partial\Psi_{01}}{\partial\theta}.\frac{\partial\Psi_{10}}{\partial\theta} + \Psi_{01}\,\nabla^2\Psi_{10} + \Psi_{10}\,\nabla^2\Psi_{01} + \gamma_{11}
\end{aligned}
$$

where the arbitrary constant γ_{11} can be shown to be zero by application of the boundary conditions. Employing the predetermined properties of $\Psi_{01}(z)$, $\Psi_{10}(\theta,z)$ and $\Psi_{11}(\theta,z)$ from (26)-(28), enables this expression to be determined as

$$
\frac{1}{\rho}p_{11}(\theta,z) = \frac{\omega_{00}}{k}\frac{\partial\Psi_{11}}{\partial z} + \left(\frac{\omega_{01}}{k} - V(z)\right)\frac{\partial\Psi_{10}}{\partial z} + \Psi_{10}\,\frac{dV}{dz}\,. \tag{35}
$$

The physical wavelike pressure component at this order is retrieved by including ε and δ to give

$$
\begin{aligned}
p\,(\theta,z) &= \varepsilon\,(p_{10} + \delta\,p_{11}) = \rho a\left[\omega_{00}.\frac{\partial\Psi_{10}}{\partial z} + \omega_{00}\frac{\partial\Psi_{11}}{\partial z} + \left(\frac{\omega_{01}}{k} - U(z)\right)\frac{\partial\Psi_{10}}{\partial z} + \Psi_{10}\,\frac{dU}{dz}\right]\\
&= \rho a\left[\left(\omega_{00} + \omega_{01} - kU(z)\right).\frac{\partial\Psi_{10}}{\partial z} + \omega_{00}\frac{\partial\Psi_{11}}{\partial z} + \Psi_{10}\,k\frac{dU}{dz}\right].
\end{aligned} \tag{36}
$$

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